LOYOLA COLLEGE (AUTONOMOUS) CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FOURTH SEMESTER – **APRIL 2025**

PMT 4501 – FUNCTIONAL ANALYSIS

Date: 23-04-2025 Dept. No. Max. : 100 Marks

Time: 01:00 PM - 04:00 PM

SECTION-A

Answer any FOUR of the following **4×10=40**

1. Let $T: X \to Y$ be a linear transformation, where X and Y are normed linear spaces. Prove that T is continuous if and only if T is bounded.
2. State and prove Holder's inequality.
3. State and prove the closed graph theorem.
4. Show that N/M is a Banach space with the defined norm when *N* is a Banach Space.
5. Prove that there exists a unique vector of smallest norm in a closed convex subset *C* of a Hilbert space *H.*
6. Show that an operator *T* on a Hilbert space *H* is self-adjoint if and only if (Tx, x) is real for all *x* in *H*.
7. Prove that if $\{e_1, e_2, e_3 \ldots . e_n\}$ is a finite orthonormal set in a Hilbert space H and $x \in H$ then (i) $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$ (ii) $x - \sum_{i=1}^{n} (x, e_i)\, e_i \perp e_j$ for each j.
8. Define the set of regular elements *G* in Banach Algebra and prove that *G* is an open set.

SECTION-B

Answer any THREE of the following **3×20=60**

9. State and prove Hahn Banach Theorem.
10. State and prove the open mapping theorem.
11. State and prove uniform boundedness principle theorem.
12. Let M be a closed subspace of a Hilbert space X, prove that every $x \in X$ has unique representation $x = y + z,\ y \in M, z \in M^{\perp}$.
13. (i) Define Banach Algebra. Let A be Banach algebra. Show that the element $x \in A$ with $\|x - 1\| < 1$ is regular then x is invertible and inverse of x is $x^{-1} = 1 + \sum_{n=1}^{\infty} (1 - x)^n$.

    (ii) Prove that the mapping $f: G \to G$ given by $f(x) = x^{-1}$ is continuous and is a homeomorphism.
14. Prove that the spectrum of an element *x* in a Banach Algebra *A* is non-empty.

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